COMPOSITE COSINE TRANSFORMS

E. OURNYCHEVA AND B. RUBIN

Dedicated to Professor Rolf Schneider on the occasion of his 65th birthday

ABSTRACT. The cosine transforms of functions on the unit sphere play an important role in convex geometry, the Banach space theory, stochastic geometry and other areas. Their higher-rank generalization to Grassmann manifolds represents an interesting mathematical object useful for applications. We introduce more general integral transforms that reveal distinctive features of higher rank objects in full generality. We call these new transforms the *composite cosine transforms*, by taking into account that their kernels agree with the composite power function of the cone of positive definite symmetric matrices. We show that injectivity of the composite cosine transforms can be studied using standard tools of the Fourier analysis on matrix spaces. In the framework of this approach, we introduce associated generalized zeta integrals and give new simple proofs to the relevant functional relations. Our technique is based on application of the higher-rank Radon transform on matrix spaces.

1. INTRODUCTION

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$, $u \cdot v$ the usual inner product of vectors $u, v \in S^{n-1}$. The classical cosine transform

$$(1.1) \qquad (Tf)(u) = \int_{S^{n-1}} f(v)|u \cdot v| \, dv, \qquad u \in S^{n-1},$$

and its generalization

$$(1.2) \qquad (T^\lambda f)(u) = \int_{S^{n-1}} f(v)|u \cdot v|^\lambda \, dv,$$

are commonly in use in convex geometry, the Banach space theory, harmonic analysis, and many other areas; see [Ga], [GH1], [Ko], [Schn]. Basic properties of T^λ (injectivity, boundedness in function spaces, and others) can be derived from the Funk-Hecke formula

$$(1.3) \qquad T^\lambda P_k = c \, \mu_k(\lambda) \, P_k,$$

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$$(1.4) \qquad c = 2\pi^{(n-1)/2}(-1)^{k/2}, \qquad \mu_k(\lambda) = \frac{\Gamma\left(\frac{\lambda+1}{2}\right)\Gamma\left(\frac{k-\lambda}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)\Gamma\left(\frac{\lambda+k+n}{2}\right)},$$

where $P_k(x)$ is the restriction of a homogeneous harmonic polynomial of even degree k [Ru].

In the last two decades a considerable attention was attracted to higher-rank generalizations of T and T^λ for functions on the Grassmann manifold $G_{n,m}$ of m-dimensional linear subspaces of $\mathbb{R}^n$. We recall that, if $\eta \in G_{n,m}$, $\xi \in G_{n,l}$, $l \geq m$, and $[\eta|\xi]$ is the m-dimensional volume of the parallelepiped spanned by the orthogonal projection of a generic orthonormal frame in η onto ξ, then, by definition,

$$(1.5) \qquad (T^\lambda f)(\xi) = \int\limits_{G_{n,m}} f(\eta)\,[\eta|\xi]^\lambda\,d\eta$$

(we adopt the same notation T^λ as in (1.2)). For $l > m$, the operator (1.5) represents the composition of the similar one over $G_{n,m}$ and the corresponding Radon transform acting from $G_{n,m}$ to $G_{n,l}$ [A], [GR]. Thus injectivity of T^λ can be studied using known results for the Radon transform (see [GR] and references therein) and the case $l = m$ in (1.5). Owing to this, in the following we assume that $l = m$, because just this case bears the basic features of the operator family (1.5).

The investigation of operators (1.5) for $\lambda = 1$ was initiated in stochastic geometry by Matheron [Mat1, p. 189], (see also [Mat2]), who conjectured that the higher-rank cosine transform T^1 is injective as well as its rank-one prototype (1.1). Matheron's conjecture was disproved in the remarkable paper by Goodey and Howard [GH1]. The higher-rank cosine transforms arise in convex geometry in the context of the generalized Shephard problem for lower dimensional projections [GZ]. More general operators T^λ for $\lambda = 0, 1, 2, \ldots$ were studied in [GH2, p. 117], where, by using reduction to $G_{4,2}$, it was proved that T^λ is non-injective for such λ. The range of the λ-cosine transform was characterized by Alesker and Bernstein [AB] ($\lambda = 1$) and by Alesker [A] (any complex λ), in terms of representations of the special orthogonal group $SO(n)$.

In this article we suggest a new approach to operators T^λ. This can be regarded as a complement to the well-known group representation method. The latter has proved to be useful in the study of Radon and cosine transforms, invariant differential and integro-differential operators on diverse homogeneous spaces of the orthogonal group; see [Goo], [Gr], [GH], [Str1], [Str2], [TT], and references therein. Our method differs from those in the cited papers. It gives a direct analog of the multiplier equality (1.3) and is applicable to a much more general operator family of the so-called *composite cosine transforms*. These are introduced in Section 3 which describes main results of the paper. In Section 4 we introduce the so-called *generalized zeta integrals* with additional "angle component" f on the relevant Stiefel manifold. An important by-product of our investigation is a functional equation for these integrals that gives rise to the composite cosine transform $T^{\boldsymbol{\lambda}}f$, for $\boldsymbol{\lambda} \in \mathbb{C}^m$, in the most general form. The case $f \equiv 1$ was studied in [FK] in the context of Jordan algebras. The argument from [FK] was extended in [Cl] when f is a determinantally homogeneous harmonic polynomial. An alternative proof of this

result for zeta integrals on matrix spaces was given in [OR2]. This proof employs an idea from [Kh] to derive the result for $\lambda \in \mathbb{C}$ as a diagonal case of the more general statement for vector-valued $\boldsymbol{\lambda} = (\lambda_1, \dots \lambda_m) \in \mathbb{C}^m$. This idea allows us to avoid essential technical difficulties (e.g., implementation of Bessel functions of matrix argument) which arise when we get stuck on the complex analysis of a single variable; cf. [FK], [Cl], [Ru4].

In the present paper, we use the same idea and suggest a new method that demonstrates application of the higher-rank Radon transform on matrix spaces [OR1], [OR3]. This Radon transform enables us to reduce the problem to the known case $f \equiv 1$. For the usual cosine transform on the unit sphere, this approach is due to A. Koldobsky [Ko]. Section 5 contains proofs of the main results.

One should note that the Fourier analysis of homogeneous distributions is one of the oldest topics in the theory of distributions, and there is a vast literature on this subject; see, e.g., [Es],[GŠ], [Le], [Ra], [Sa], [Se].

Acknowledgement. We are thankful to Professors S. Alesker, P. Goodey and W. Weil for very helpful comments and pleasant discussion.

2. Preliminaries

The main references for this section are [FK], [Gi], [T].

2.1. **Notation.** Let $\mathbb{R}^{n \times m}$ be the space of real matrices $x = (x_{i,j})$ having n rows and m columns; $dx = \prod_{i=1}^{n} \prod_{j=1}^{m} dx_{i,j}$. In the following, x' denotes the transpose of x, I_m is the identity $m \times m$ matrix, and $O(n)$ is the group of real orthogonal $n \times n$ matrices. For $n \geq m$, we denote by $V_{n,m} = \{v \in \mathbb{R}^{n \times m} : v'v = I_m\}$ the Stiefel manifold of orthonormal m-frames in $\mathbb{R}^n$. This is a homogeneous space with respect to the action $V_{n,m} \ni v \to \gamma v$, $\gamma \in O(n)$, so that $V_{n,m} = O(n)/O(n - m)$. The invariant measure dv on $V_{n,m}$ induced by the Lebesgue measure on the ambient space is defined up to a constant multiple. We normalize it using geometric argument and set

$$(2.1) \qquad \sigma_{n,m} \equiv \int_{V_{n,m}} dv = \prod_{i=1}^{m} |S^{n-i}|,$$

where $|S^i| = 2\pi^{(i+1)/2}/\Gamma((i + 1)/2)$ is the surface area of the i-dimensional unit sphere.

Let Ω be the cone of positive definite symmetric matrices $r = (r_{i,j})_{m \times m}$ with the elementary volume $dr = \prod_{i \leq j} dr_{i,j}$. We denote $|r| = \det(r)$ and let $d_*r = |r|^{-(m+1)/2}dr$ be the $GL(m, \mathbb{R})$-invariant measure on Ω. If T_m is the group of upper triangular $m \times m$ matrices $t = (t_{i,j})$ with positive diagonal elements, then each $r \in \Omega$ has a unique representation $r = t't$.

We will constantly use the polar coordinates and the spherical coordinates on $\mathbb{R}^{n \times m}$ which are defined as follows.

Lemma 2.1. ([Mu, pp. 66, 591], [Ma]) *If* $x \in \mathbb{R}^{n \times m}$, $\mathrm{rank}(x) = m$, $n \geq m$, *then*

$$(2.2) \qquad x = vr^{1/2}, \qquad v \in V_{n,m}, \qquad r = x'x \in \Omega,$$

and $dx = 2^{-m}|r|^{(n-m-1)/2}drdv.$

Lemma 2.2. ([P], [Ru4]) *If* $x \in \mathbb{R}^{n \times m}$, $\mathrm{rank}(x) = m$, $n \geq m$, *then*

$$x = ut, \qquad u \in V_{n,m}, \qquad t \in T_m,$$

and

$$dx = \prod_{j=1}^{m} t_{j,j}^{n-j} \, dt_{j,j} \, dt_* dv, \qquad dt_* = \prod_{i<j} dt_{i,j}.$$

The Schwartz space $S = S(\mathbb{R}^{n \times m})$ is identified with the respective space on $\mathbb{R}^{nm}$ of infinitely differentiable rapidly decreasing functions. The Fourier transform of a function $f \in L^1(\mathbb{R}^{n \times m})$ is defined by

$$(2.3) \qquad (\mathcal{F}f)(y) = \int_{\mathbb{R}^{n \times m}} e^{\text{tr}(iy'\text{x})} f(x) dx, \qquad y \in \mathbb{R}^{n \times m} .$$

The relevant Parseval equality reads

$$(2.4) \qquad (\mathcal{F}f, \mathcal{F}\varphi) = (2\pi)^{nm} (f, \varphi), \qquad (f, \varphi) = \int_{\mathbb{R}^{n \times m}} f(x) \overline{\varphi(x)} \, dx.$$

2.2. **The composite power function.** Given $r = (r_{i,j}) \in \Omega$, let $\Delta_0(r) = 1$, $\Delta_1(r) = r_{1,1}$, $\Delta_2(r)$, ..., $\Delta_m(r) = |r|$ be the corresponding principal minors which are strictly positive. For $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m$, the composite power function of the cone Ω is defined by

$$(2.5) \qquad r^{\boldsymbol{\lambda}} = \prod_{i=1}^{m} \left[\frac{\Delta_i(r)}{\Delta_{i-1}(r)} \right]^{\lambda_i/2} = \Delta_1(r)^{\frac{\lambda_1 - \lambda_2}{2}} \ldots \Delta_{m-1}(r)^{\frac{\lambda_{m-1} - \lambda_m}{2}} \Delta_m(r)^{\frac{\lambda_m}{2}} .$$

In the special case $\lambda_1 = \ldots = \lambda_m = \lambda$ we write $\boldsymbol{\lambda}_0 = (\lambda, \ldots, \lambda)$ $(\in \mathbb{C}^m)$ so that $r^{\boldsymbol{\lambda}_0} = |r|^{\lambda/2}$. If $r = t't$, $t = (t_{i,j}) \in T_m$, then $r^{\boldsymbol{\lambda}} = \prod_{j=1}^{m} t_{j,j}^{\lambda_j}$. This implies the following equalities:

$$(2.6) \qquad r^{\boldsymbol{\lambda}+\boldsymbol{\mu}} = r^{\boldsymbol{\lambda}} r^{\boldsymbol{\mu}}, \quad r^{\boldsymbol{\lambda}+\boldsymbol{\alpha}_0} = r^{\boldsymbol{\lambda}} |r|^{\alpha/2}, \quad \boldsymbol{\alpha}_0 = (\alpha, \ldots, \alpha);$$

$$(2.7) \qquad (t'rt)^{\boldsymbol{\lambda}} = (t't)^{\boldsymbol{\lambda}} r^{\boldsymbol{\lambda}}, \quad t \in T_m.$$

The reverses of $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ and $r = (r_{i,j}) \in \Omega$ are defined by

$$\boldsymbol{\lambda}_* = (\lambda_m, \ldots, \lambda_1); \qquad r_* = \omega r \omega, \qquad \omega = \begin{bmatrix} 0 & & & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & & & 0 \end{bmatrix},$$

so that

$$(\boldsymbol{\lambda}_*)_j = \lambda_{m-j+1}, \qquad (r_*)_{i,j} = r_{m-i+1,m-j+1}.$$

We have

$$(2.8) \qquad r^{\boldsymbol{\lambda}_*} = (r^{-1})_*^{-\boldsymbol{\lambda}}, \qquad (r^{-1})^{\boldsymbol{\lambda}} = r_*^{-\boldsymbol{\lambda}_*}.$$

The gamma function of the cone Ω is defined by

$$(2.9) \qquad \Gamma_\Omega(\boldsymbol{\lambda}) = \int_\Omega r^{\boldsymbol{\lambda}} e^{-\text{tr}(r)} d_* r = \pi^{m(m-1)/4} \prod_{j=1}^{m} \Gamma((\lambda_j - j + 1)/2);$$

see, e.g., [FK, p. 123]. The integral in (2.9) converges absolutely if and only if $Re \, \lambda_j > j - 1$ for all $j = 1, \ldots, m$, and extends meromorphically to all $\lambda \in \mathbb{C}^m$. The following relation holds:

$$(2.10) \qquad \int_\Omega r^{\boldsymbol{\lambda}} e^{-\text{tr}(rs)} d_* r = \Gamma_\Omega(\boldsymbol{\lambda}) s_*^{-\boldsymbol{\lambda}_*}, \qquad s \in \Omega.$$

An important particular case of (2.9) is the Siegel integral

$$(2.11) \quad \Gamma_m(\lambda) = \int_\Omega |r|^\lambda e^{-\mathrm{tr}(r)} d_* r = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(\lambda - j/2), \quad Re\,\lambda > (m-1)/2.$$

If $\boldsymbol{\lambda}_0 = (\lambda, \ldots, \lambda)$, then $\Gamma_\Omega(\boldsymbol{\lambda}_0) = \Gamma_m(\lambda/2)$. The volume $\sigma_{n,m}$ of the Stiefel manifold $V_{n,m}$ may be written in terms of the Siegel Gamma function:

$$(2.12) \qquad\qquad \sigma_{n,m} = \frac{2^m \pi^{nm/2}}{\Gamma_m(n/2)}.$$

2.3. **Radon transforms on the space of matrices.** The main references for this subsection are [OR1], [OR3], [OR4], [P], [Sh1], [Sh2]. We fix positive integers k, n, and m, $0 < k < n$, and let $V_{n,k}$ be the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$. For $\xi \in V_{n,k}$ and $t \in \mathbb{R}^{k \times m}$, the linear manifold

$$(2.13) \qquad\qquad \tau = \tau(\xi, t) = \{x \in \mathbb{R}^{n \times m} : \xi' x = t\}$$

is called a *matrix $(n-k)$-plane* in $\mathbb{R}^{n \times m}$. We denote by $\mathfrak{T}$ the set of all such planes. Each $\tau \in \mathfrak{T}$ is an ordinary $(n-k)m$-dimensional plane in $\mathbb{R}^{nm}$, but the set $\mathfrak{T}$ has measure zero in the manifold of all such planes. The *matrix Radon transform* $f(x) \to (\mathcal{R}_k f)(\tau)$ assigns to a function $f(x)$ on $\mathbb{R}^{n \times m}$ a collection of integrals of f over all matrix planes $\tau \in \mathfrak{T}$, namely,

$$(\mathcal{R}_k f)(\tau) = \int_{x \in \tau} f(x).$$

Precise meaning of this integral is the following:

$$(2.14) \qquad\qquad (\mathcal{R}_k f)(\tau) = \int_{\mathbb{R}^{(n-k) \times m}} f\left(g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix}\right) d\omega,$$

where $g_\xi \in SO(n)$ is a rotation satisfying

$$(2.15) \qquad\qquad g_\xi \xi_0 = \xi, \qquad \xi_0 = \begin{bmatrix} 0 \\ I_k \end{bmatrix} \in V_{n,k}.$$

The following statement is a matrix generalization of the so-called projection-slice theorem. It links together the Fourier transform (2.3) and the Radon transform (2.14). In the case $m = 1$, this theorem can be found in [Na, p. 11] (for $k = 1$) and [Ke, p. 283] (for any $0 < k < n$).

Theorem 2.3. ([OR4]) *For $f \in L^1(\mathbb{R}^{n \times m})$ and $1 \leq m \leq k$,*

$$(2.16) \qquad (\mathcal{F}f)(\xi b) = [\tilde{\mathcal{F}}(\mathcal{R}_k f)(\xi, \cdot)](b), \quad \xi \in V_{n,k}, \quad b \in \mathbb{R}^{k \times m}.$$

Here, $\tilde{\mathcal{F}}\varphi$ denotes the Fourier transform of a function $t \to \varphi(\xi, t)$ on the space $\mathbb{R}^{k \times m}$.

3. Main results

For $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m$, we introduce intertwining operators

$$(3.1) \qquad (T^{\boldsymbol{\lambda}} f)(u) = \int_{V_{n,m}} f(v)\, (u'vv'u)^{\boldsymbol{\lambda}}\, dv, \qquad u \in V_{n,m}, \qquad n > m,$$

that commute with the left action of $O(n)$. We call $T^{\boldsymbol{\lambda}} f$ *the composite cosine transform of* f. If $\lambda_1 = \ldots = \lambda_m = \lambda$, then (3.1) reads

$$(3.2) \qquad (T^\lambda f)(u) = \int\limits_{V_{n,m}} f(v)|\det(v'u)|^\lambda \, dv.$$

If f is a $O(m)$ right-invariant function on $V_{n,m}$, then (3.2) can be identified with (1.5) (for $l = m$) and represents the usual λ-cosine transform on $G_{n,m}$.

Definition 3.1. We denote by $\mathfrak{L}$ the set of all $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m$ satisfying $Re \, \lambda_j > j - m - 1$ for all $j = 1, \ldots, m$.

This definition is motivated by the following.

Theorem 3.2. *For $f \in L^1(V_{n,m})$, the integral $(T^{\boldsymbol{\lambda}} f)(u)$ converges absolutely a.e. on $V_{n,m}$ if and only if $\boldsymbol{\lambda} \in \mathfrak{L}$, and represents an analytic function of $\boldsymbol{\lambda}$ in this domain. For such $\boldsymbol{\lambda}$, the linear operator $T^{\boldsymbol{\lambda}}$ is bounded on $L^1(V_{n,m})$.*

This statement follows immediately by Fubini's theorem from the equality [OR2]

$$(3.3) \qquad \int\limits_{V_{n,m}} (u'vv'u)^{\boldsymbol{\lambda}} \, du = \frac{2^m \, \pi^{nm/2}}{\Gamma_m(m/2)} \frac{\Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{m}_0)}{\Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{n}_0)} \quad (\equiv T^{\boldsymbol{\lambda}} 1)$$

which is of independent interest.

It is challenging to describe the set of all $\boldsymbol{\lambda} \in \mathbb{C}^m$ for which $T^{\boldsymbol{\lambda}}$ is injective. We cannot solve this problem in full generality and restrict our consideration to the space $L^\flat(V_{n,m})$ of $O(m)$ right-invariant integrable functions on $V_{n,m}$. This allows us to obtain a precise description of those $\boldsymbol{\lambda}$ for which $T^{\boldsymbol{\lambda}}$ is injective in the following important cases (a) $2m \leq n$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m$, and (b) $\lambda_1 = \cdots = \lambda_m = \lambda \in \mathbb{C}$, provided that $T^{\boldsymbol{\lambda}} f$ and $T^\lambda f$ exist as absolutely convergent integrals. Note that for $m = 1$, the space $L^\flat(V_{n,m})$ is actually the space of even integrable functions on the unit sphere.

Theorem 3.3. *Let $n > m \geq 1$ and $\boldsymbol{\lambda} \in \mathfrak{L}$, i.e. $Re \, \lambda_j > j - m - 1$ for all $j = 1, \ldots, m$. If, moreover,*

$$(3.4) \qquad \lambda_j + m - j \neq 0, 2, 4, \ldots \quad \text{for all} \quad j = 1, \ldots, m,$$

then the composite cosine transform $T^{\boldsymbol{\lambda}}$ is injective on $L^\flat(V_{n,m})$. If $2m \leq n$ and (3.4) fails, then $T^{\boldsymbol{\lambda}}$ is non-injective. Specifically, it annihilates all $O(m)$ right-invariant, harmonic, determinantally homogeneous polynomials of degree $k > \max\limits_j \{\lambda_j + m - j\}$ (see Definition 5.1).

Some comments are in order. The essence of our approach is that we apply the standard Fourier transform technique to obtain a higher-rank analog of (1.3). We do not know if the condition (3.4) is necessary for injectivity of $T^{\boldsymbol{\lambda}}$ when $2m > n$. To answer this question, one has to treat $T^{\boldsymbol{\lambda}}$ on polynomial representations of $SO(n)$. These are parameterized by highest weights $(m_1, m_2, \ldots, m_{[n/2]})$ that are more general than those adopted in the framework of our approach; cf. [Str1], [TT]. However, if $\lambda_1 = \cdots = \lambda_m = \lambda$, then for the λ-cosine transform (1.5), we give the following complete answer which reveals essential difference between the rank-one case and that of a higher-rank.

Theorem 3.4. *Let $n > m$, $Re\,\lambda > -1$, and let $r_{n,m} = \mathrm{rank}(G_{n,m}) = \min(m, n-m)$ be the rank of the Grassmannian $G_{n,m}$. If $r_{n,m} = 1$, then T^λ is injective on $L^1(G_{n,m})$ if and only if $\lambda \neq 0, 2, 4, \ldots$. If $r_{n,m} > 1$, then T^λ is injective on $L^1(G_{n,m})$ if and only if $\lambda \neq 0, 1, 2, \ldots$.*

This statement is known. It follows from the more general result of Alesker [A].

4. The generalized zeta integrals and the composite cosine transforms

4.1. The generalized zeta integrals. In accordance with the polar decomposition $x = vr^{1/2}$, $v \in V_{n,m}$, $r = x'x \in \Omega$, we introduce *the generalized zeta integrals* (or zeta distributions):

$$(4.1) \qquad \mathcal{Z}(\phi, \boldsymbol{\lambda}, f) = \int\limits_{\mathbb{R}^{n \times m}} r^{\boldsymbol{\lambda}}\, f(v)\, \overline{\phi(x)}\, dx = (r^{\boldsymbol{\lambda}} f, \phi),$$

$$(4.2) \qquad \mathcal{Z}_*(\phi, \boldsymbol{\lambda}, f) = \int\limits_{\mathbb{R}^{n \times m}} r_*^{\boldsymbol{\lambda}}\, f(v)\, \overline{\phi(x)}\, dx = (r_*^{\boldsymbol{\lambda}} f, \phi),$$

where $f \in L^1(V_{n,m})$ and $\phi \in S(\mathbb{R}^{n \times m})$. Zeta integrals of this type with the angle component $f \equiv 1$ are well known and arise in different occurrences; see [FK], [BSZ]) and references therein. We denote

$$(4.3) \qquad \boldsymbol{\Lambda} = \{\boldsymbol{\lambda} \in \mathbb{C}^m : Re\,\lambda_j > j - n - 1 \quad \text{for all} \quad j = 1, \ldots, m\},$$

$$(4.4) \qquad \boldsymbol{\Lambda}_0 = \{\boldsymbol{\lambda} \in \mathbb{C}^m : \lambda_j = j - n - l \quad \text{for some}$$
$$j \in \{1, \ldots, m\}, \quad \text{and} \quad l \in \{1, 3, 5, \ldots\}\}.$$

Lemma 4.1. *The integrals (4.1) and (4.2) are absolutely convergent if and only if $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$, and extend as meromorphic functions of $\boldsymbol{\lambda}$ with the polar set $\boldsymbol{\Lambda}_0$. The normalized zeta integrals*

$$(4.5) \qquad \mathcal{Z}^0(\phi, \boldsymbol{\lambda}, f) = \frac{\mathcal{Z}(\phi, \boldsymbol{\lambda}, f)}{\Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{n}_0)}, \qquad \mathcal{Z}_*^0(\phi, \boldsymbol{\lambda}, f) = \frac{\mathcal{Z}_*(\phi, \boldsymbol{\lambda}, f)}{\Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{n}_0)},$$

$\mathbf{n}_0 = (n, \ldots, n)$, *are entire functions of $\boldsymbol{\lambda}$.*

To prove this lemma it suffices to write both integrals in spherical coordinates and then apply a standard argument from [GSh]; see [OR2] for details.

4.2. The basic functional equation. The connection between zeta integrals and composite cosine transforms can be established in the form of a functional equation which is actually the usual Parseval equality. Note that the function $(T^{\boldsymbol{\lambda}} f)(u)$, initially defined for $u \in V_{n,m}$, extends to all matrices $y \in \mathbb{R}^{n \times m}$ of rank m. Indeed, by (2.7), in spherical coordinates $y = ut$, $u \in V_{n,m}$, $t \in T_m$, we have

$$(4.6) \qquad (T^{\boldsymbol{\lambda}} f)(y) = r^{\boldsymbol{\lambda}} (T^{\boldsymbol{\lambda}} f)(u)$$

where $r^{\boldsymbol{\lambda}} = (t't)^{\boldsymbol{\lambda}} = (y'y)^{\boldsymbol{\lambda}}$ is the "radial part" of $(T^{\boldsymbol{\lambda}} f)(y)$.

Theorem 4.2. *Let*

$$(4.7) \qquad \varphi_{\boldsymbol{\lambda}}(x) = \frac{r_*^{-\boldsymbol{\lambda}_* - \mathbf{n}_0}}{\Gamma_\Omega(-\boldsymbol{\lambda}_*)}\, f(v), \qquad x = vr^{1/2}, \quad \mathbf{n}_0 = (n, \ldots, n), \quad \boldsymbol{\lambda} \in \mathbb{C}^m.$$

If f is an integrable $O(m)$ right-invariant function on $V_{n,m}$, then

$$(4.8) \qquad (\mathcal{F}\varphi_{\boldsymbol{\lambda}})(y) = \frac{c_{\boldsymbol{\lambda}}}{\Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{m}_0)}\,(T^{\boldsymbol{\lambda}}f)(y), \qquad c_{\boldsymbol{\lambda}} = 2^{-|\boldsymbol{\lambda}|}\pi^{m^2/2}/\sigma_{m,m},$$

in the sense of S'-distributions. In other words, for each $\phi \in S(\mathbb{R}^{n\times m})$,

$$(4.9) \quad \frac{c_{\boldsymbol{\lambda}}}{\Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{m}_0)}\,(T^{\boldsymbol{\lambda}}f,\ \mathcal{F}\phi) = (2\pi)^{nm}\,(\varphi_{\boldsymbol{\lambda}},\ \phi) \equiv (2\pi)^{nm}\,\mathcal{Z}^0_*(\phi, -\boldsymbol{\lambda}_* - \mathbf{n}_0, f).$$

A self-contained proof of this statement is given in [OR2]. Here, we apply an alternative approach which is of independent interest. The main idea is to reduce the problem to the corresponding functional equation containing zeta integrals on $\mathbb{R}^{m\times m}$ with the angle component $f \equiv 1$. To this end, we invoke the higher-rank Radon transform (2.14). We start with two auxiliary lemmas.

Lemma 4.3. *For $r \in \Omega$ and $y \in \mathbb{R}^{n\times m}$,*

$$(4.10) \qquad\qquad \mathcal{F}\Big[\frac{r_*^{-\boldsymbol{\lambda}_*-\mathbf{n}_0}}{\Gamma_\Omega(-\boldsymbol{\lambda}_*)}\Big](y) = \frac{2^{-|\boldsymbol{\lambda}|}\pi^{nm/2}}{\Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{n}_0)}\,(y'y)^{\boldsymbol{\lambda}}$$

in the sense of S'-distributions.

Proof. Formula (4.10) was established by Khekalo [Kh] who modified the argument from [St2, Chapter III, Sec. 3.4] for functions of matrix argument; see also [FK], [OR4], [Ru4] on this subject. For convenience of the reader, we outline the proof of (4.10) in our notation. Since

$$\mathcal{F}[e^{-\mathrm{tr}(xsx'/4\pi)}](y) = (2\pi)^{nm}|s|^{-n/2}e^{-\mathrm{tr}(\pi ys^{-1}y')},$$

for $s \in \Omega$ and $\phi \in S(\mathbb{R}^{n\times m})$, the Parseval equality yields

$$(4.11) \qquad |s|^{-n/2}\int_{\mathbb{R}^{n\times m}} e^{-\mathrm{tr}(\pi ys^{-1}y')}\,\overline{(\mathcal{F}\phi)(y)}\,dy = \int_{\mathbb{R}^{n\times m}} e^{-\mathrm{tr}(xsx'/4\pi)}\,\overline{\phi(x)}\,dx.$$

We multiply (4.11) by $s^{\boldsymbol{\lambda}+\mathbf{n}_0}$ and integrate against d_*s. This gives

$$\int_{\mathbb{R}^{n\times m}} I_1(y)\,\overline{(\mathcal{F}\phi)(y)}\,dy = \int_{\mathbb{R}^{n\times m}} I_2(x)\,\overline{\phi(x)}\,dx,$$

where

$$I_1(y) = \int_\Omega s^{\boldsymbol{\lambda}}e^{-\mathrm{tr}(\pi ys^{-1}y')}d_*s, \quad I_2(x) = \int_\Omega s^{\boldsymbol{\lambda}+\mathbf{n}_0}e^{-\mathrm{tr}(xsx'/4\pi)}d_*s.$$

Evaluation of the last integrals by means of (2.10) gives

$$I_1(y) = \pi^{|\boldsymbol{\lambda}|/2}\Gamma_\Omega(-\boldsymbol{\lambda}_*)(y'y)^{\boldsymbol{\lambda}}, \quad \mathrm{Re}\,\lambda_j < j - m,$$

and

$$I_2(x) = (4\pi)^{(|\boldsymbol{\lambda}|+nm)/2}\Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{n}_0)(x'x)_*^{-\boldsymbol{\lambda}_*-\mathbf{n}_0}, \quad \mathrm{Re}\,\lambda_j > j - n - 1.$$

Hence, if $j - n - 1 < \mathrm{Re}\,\lambda_j < j - m$, then

$$(4.12)\ \Gamma_\Omega(-\boldsymbol{\lambda}_*)\int_{\mathbb{R}^{n\times m}} (y'y)^{\boldsymbol{\lambda}}\,\overline{(\mathcal{F}\phi)(y)}\,dy = c_{\boldsymbol{\lambda}}\Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{n}_0)\int_{\mathbb{R}^{n\times m}} (x'x)_*^{-\boldsymbol{\lambda}_*-\mathbf{n}_0}\,\overline{\phi(x)}\,dx,$$

where $c_{\boldsymbol{\lambda}} = 2^{nm+|\boldsymbol{\lambda}|}\pi^{nm/2}$. By Lemma 4.1, this extends analytically to all $\boldsymbol{\lambda}$, and we are done. $\qquad\square$

Lemma 4.4. *For $\phi \in S(\mathbb{R}^{n \times m})$ and $\xi \in V_{n,m}$,*

$$(4.13) \quad \frac{1}{\Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{m}_0)} \int\limits_{\mathbb{R}^{n \times m}} (x'\xi\xi'x)^{\boldsymbol{\lambda}} \, \overline{(\mathcal{F}\phi)(x)} dx = \frac{d_{\boldsymbol{\lambda}}}{\Gamma_\Omega(-\boldsymbol{\lambda}_*)} \int\limits_{\mathbb{R}^{m \times m}} (t't)_*^{-\boldsymbol{\lambda} - \mathbf{m}_0} \overline{\phi(\xi t)} dt,$$

$$(4.14) \qquad\qquad\qquad\qquad d_{\boldsymbol{\lambda}} = 2^{mn + |\boldsymbol{\lambda}|} \pi^{nm - m^2/2}.$$

Proof. Let us denote by $A(\xi)$ the left side of (4.13). By passing to the spherical coordinates on $\mathbb{R}^{n \times m}$, according to Lemma 2.2 and (2.7) we obtain

$$A(\xi) = \frac{1}{\Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{m}_0)} \int\limits_{T_m} \prod_{j=1}^m t_{j,j}^{\lambda_j + n - j} dt_{j,j} dt_* \int\limits_{V_{n,m}} \overline{(\mathcal{F}\phi)(ut)} \, (u'\xi\xi'u)^{\boldsymbol{\lambda}} \, du.$$

By Theorem 3.2, this integral converges absolutely if $\boldsymbol{\lambda} \in \mathfrak{L}$, that is $Re\,\lambda_j > j - m - 1$ for all $j = 1, \ldots, m$. Let us evaluate $A(\xi)$ for such $\boldsymbol{\lambda}$.

We put $x = g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix}$, where $g_\xi \in SO(n)$ is a rotation satisfying (2.15) with $k = m$, $\omega \in \mathbb{R}^{(n-m) \times m}$, and $t \in \mathbb{R}^{m \times m}$. Then $\xi'x = t$, so that the Fubini theorem and (4.10) yield

$$\begin{aligned}
A(\xi) &= \frac{1}{\Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{m}_0)} \int\limits_{\mathbb{R}^{m \times m}} (t't)^{\boldsymbol{\lambda}} \, dt \int\limits_{\mathbb{R}^{(n-m) \times m}} \overline{(\mathcal{F}\phi)(g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix})} d\omega \\[2mm]
&= \frac{1}{\Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{m}_0)} \int\limits_{\mathbb{R}^{m \times m}} (t't)^{\boldsymbol{\lambda}} \overline{(\mathcal{R}_m \mathcal{F}\phi)(\xi, t)} \, dt \\[2mm]
&= \frac{2^{|\boldsymbol{\lambda}| + m^2} \pi^{m^2/2}}{\Gamma_\Omega(-\boldsymbol{\lambda}_*)} \int\limits_{\mathbb{R}^{m \times m}} (t't)_*^{-\boldsymbol{\lambda}_* - \mathbf{m}_0} \overline{(\tilde{\mathcal{F}}^{-1} \mathcal{R}_m \mathcal{F}\phi)(\xi, t)} \, dt \\[2mm]
&= \frac{2^{|\boldsymbol{\lambda}|} \pi^{-m^2/2}}{\Gamma_\Omega(-\boldsymbol{\lambda}_*)} \int\limits_{\mathbb{R}^{m \times m}} (t't)_*^{-\boldsymbol{\lambda}_* - \mathbf{m}_0} \overline{(\tilde{\mathcal{F}} \mathcal{R}_m \mathcal{F}\phi)(\xi, t)} \, dt.
\end{aligned}$$

Here, $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^{-1}$ denote the Fourier transform acting in the t-variable on the space $\mathbb{R}^{m \times m}$, and its inverse, respectively. We have used the equality $(\tilde{\mathcal{F}}^{-1}\varphi)(\xi, t) = (2\pi)^{-m^2}(\tilde{\mathcal{F}}\varphi)(\xi, -t)$. Moreover, we have applied (4.10) with $n = m$ in the case when both functionals in this equality are regular. The latter is true if

$$(4.15) \qquad\qquad\qquad j - m - 1 < Re\,\lambda_j < j - m.$$

By (2.16),

$$\tilde{\mathcal{F}}[\mathcal{R}_m \mathcal{F}\phi(\xi, \cdot)](t) = (\mathcal{F}\mathcal{F}\phi)(\xi t) = (2\pi)^{nm} \phi(-\xi t).$$

This gives

$$A(\xi) = \frac{2^{|\boldsymbol{\lambda}| + nm} \pi^{nm - m^2/2}}{\Gamma_\Omega(-\boldsymbol{\lambda}_*)} \int\limits_{\mathbb{R}^{m \times m}} (t't)_*^{-\boldsymbol{\lambda}_* - \mathbf{m}_0} \overline{\phi(\xi t)} \, dt.$$

Therefore, (4.13) is valid when $\boldsymbol{\lambda}$ satisfies (4.15). By Lemma 4.1, this extends analytically to all $\boldsymbol{\lambda} \in \mathbb{C}^m$. $\qquad\qquad\square$

Proof of Theorem 4.2. Let $\boldsymbol{\lambda}$ obey (4.15), so that by Theorem 3.2 and Lemma 4.1, integrals in (4.9) are absolutely convergent. Then, by (4.13) and the Fubini theorem,

$$
\begin{aligned}
\frac{1}{\Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{m}_0)}(T^{\boldsymbol{\lambda}}f,\ \mathcal{F}\phi) &= \frac{1}{\Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{m}_0)}\int\limits_{\mathbb{R}^{n\times m}}(T^{\boldsymbol{\lambda}}f)(x)\overline{(\mathcal{F}\phi)(x)}dx \\
&= \frac{1}{\Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{m}_0)}\int\limits_{V_{n,m}}f(v)dv\int\limits_{\mathbb{R}^{n\times m}}\overline{(\mathcal{F}\phi)(x)}(x'vv'x)^{\boldsymbol{\lambda}}\,dx \\
&= \frac{d_{\boldsymbol{\lambda}}}{\Gamma_\Omega(-\boldsymbol{\lambda}_*)}\int\limits_{V_{n,m}}f(v)dv\int\limits_{\mathbb{R}^{m\times m}}(t't)_*^{-\boldsymbol{\lambda}_*-\mathbf{m}_0}\overline{\phi(vt)}dt.
\end{aligned}
$$

Here, $d_{\boldsymbol{\lambda}}$ is defined by (4.14). By the polar decomposition (see Lemma 2.1), for the right-invariant function f we obtain

$$
\begin{aligned}
\frac{1}{\Gamma_\Omega(\boldsymbol{\lambda}+\mathbf{m}_0)}(T^{\boldsymbol{\lambda}}f,\ \mathcal{F}\phi) &= \frac{2^{-m}d_{\boldsymbol{\lambda}}\sigma_{m,m}}{\Gamma_\Omega(-\boldsymbol{\lambda}_*)}\int\limits_{V_{n,m}}f(v)dv\int\limits_{O(m)}d\gamma\int\limits_{\Omega}r_*^{-\boldsymbol{\lambda}_*-\mathbf{m}_0}|r|^{-1/2}\overline{\phi(v\gamma r^{1/2})}dr \\
&= \frac{2^{-m}d_{\boldsymbol{\lambda}}\sigma_{m,m}}{\Gamma_\Omega(-\boldsymbol{\lambda}_*)}\int\limits_{V_{n,m}}f(v)dv\int\limits_{\Omega}r_*^{-\boldsymbol{\lambda}_*-\mathbf{m}_0}|r|^{-1/2}\overline{\phi(vr^{1/2})}dr \\
&= \frac{d_{\boldsymbol{\lambda}}\sigma_{m,m}}{\Gamma_\Omega(-\boldsymbol{\lambda}_*)}\int\limits_{\mathbb{R}^{n\times m}}f(x(x'x)^{-1/2})(x'x)_*^{-\boldsymbol{\lambda}_*-\mathbf{n}_0}\overline{\phi(x)}dx.
\end{aligned}
$$

Hence, (4.9) is proved for $j-m-1 < Re\,\lambda_j < j-m$. According to Lemma 4.1, this extends analytically to all $\boldsymbol{\lambda} \in \mathbb{C}^m$.

Example 4.5. Let $m \geq 1$, $\lambda_1 = \ldots = \lambda_m = \lambda$, $|x|_m = \det(x'x)^{1/2}$. Then

$$
\varphi_{\boldsymbol{\lambda}}(x) = \frac{|x|_m^{-\lambda-n}}{\Gamma_m(-\lambda/2)}\,f(x(x'x)^{-1/2}), \qquad (T^{\lambda}f)(y) = \int\limits_{V_{n,m}}f(v)|\det(v'y)|^{\lambda}\,dv.
$$

If f is an integrable $O(m)$ right-invariant function on $V_{n,m}$, then the operator T^{λ} coincides with (1.5), and we have

$$
(4.16) \qquad (\mathcal{F}\varphi_{\boldsymbol{\lambda}})(y) = \frac{2^{-\lambda m}\,\pi^{m^2/2}}{\sigma_{m,m}\Gamma_m((\lambda+m)/2)}\,(T^{\lambda}f)(y).
$$

If $m=1$, and $x,y \in \mathbb{R}^n \setminus \{0\}$, then

$$
\varphi_{\boldsymbol{\lambda}}(x) = \frac{|x|^{-\lambda-n}}{\Gamma(-\lambda/2)}\,f\Big(\frac{x}{|x|}\Big), \qquad (T^{\lambda}f)(y) = \int\limits_{S^{n-1}}f(v)|v\cdot y|^{\lambda}\,dv.
$$

In this case,

$$
(4.17) \qquad (\mathcal{F}\varphi_{\boldsymbol{\lambda}})(y) = \frac{2^{-1-\lambda}\pi^{1/2}}{\Gamma((\lambda+1)/2)}\,(T^{\lambda}f)(y).
$$

The last equality is well known and can be found in many sources.

4.3. **The case of homogeneous polynomials.** Let $P_k(x)$ be a polynomial on $\mathbb{R}^{n \times m}$ which is harmonic (as a function on $\mathbb{R}^{nm}$) and determinantally homogeneous of degree k, i.e., $P_k(xg) = \det(g)^k P_k(x)$, $\forall g \in GL(m, \mathbb{R})$. It means that P_k is a usual homogeneous harmonic polynomial of degree km on $\mathbb{R}^{nm}$. Theorem 4.2 can be strengthened if we choose f to be the restriction of $P_k(x)$ onto $V_{n,m}$.

Theorem 4.6. *Let*

$$(4.18) \qquad \varphi_{\boldsymbol{\lambda},k}(x) = \frac{r_*^{-\boldsymbol{\lambda}_* - \mathbf{n}_0}}{\Gamma_\Omega(-\boldsymbol{\lambda}_*)} P_k(v), \qquad x = vr^{1/2}, \quad d_{\boldsymbol{\lambda}} = 2^{-|\boldsymbol{\lambda}|} \pi^{nm/2} i^{km}.$$

Then, for all $\boldsymbol{\lambda} \in \mathbb{C}^m$,

$$(4.19) \qquad (\mathcal{F}\varphi_{\boldsymbol{\lambda},k})(vr^{1/2}) = \frac{d_{\boldsymbol{\lambda}} \, \Gamma_\Omega(\mathbf{k}_0 - \boldsymbol{\lambda}_*)}{\Gamma_\Omega(-\boldsymbol{\lambda}_*) \, \Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{k}_0 + \mathbf{n}_0)} P_k(v) \, r^{\boldsymbol{\lambda}}$$

in the sense of S'-distributions. In other words, for each $\phi \in S(\mathbb{R}^{n \times m})$,

$$\frac{d_{\boldsymbol{\lambda}} \, \Gamma_\Omega(\mathbf{k}_0 - \boldsymbol{\lambda}_*)}{\Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{k}_0 + \mathbf{n}_0)} \, (P_k(v) \, r^{\boldsymbol{\lambda}}, \, \mathcal{F}\phi) \;\; = \;\; (2\pi)^{nm} (\varphi_{\boldsymbol{\lambda},k}, \, \phi)$$

$$= \;\; (2\pi)^{nm} \, \mathcal{Z}_*^0(\phi, -\boldsymbol{\lambda}_* - \mathbf{n}_0, P_k).$$

Proof. The classical Hecke identity

$$\int_{\mathbb{R}^{n \times m}} P_k(x) e^{-\mathrm{tr}(\pi x'x)} e^{\mathrm{tr}(2\pi i y'x)} \, dx = i^{km} P_k(y) e^{-\mathrm{tr}(\pi y'y)}$$

implies that

$$(4.20) \qquad |s|^{-k-n/2} \int_{\mathbb{R}^{n \times m}} P_k(y) e^{-\mathrm{tr}(\pi y s^{-1} y')} \, \overline{(\mathcal{F}\phi)(y)} \, dy$$

$$= \;\; (2\pi i)^{-km} \int_{\mathbb{R}^{n \times m}} P_k(x) e^{-\mathrm{tr}(xsx'/4\pi)} \, \overline{\phi(x)} \, dx.$$

We multiply this by $s^{\boldsymbol{\lambda} + \mathbf{n}_0 + \mathbf{k}_0}$ and proceed, as in the proof of Lemma 4.3, to obtain the result for $j - n - k - 1 < Re \, \lambda_j < j + k - m$. Since this domain is not void for all $k = 0, 1, 2, \ldots$, and the normalized zeta integral $\mathcal{Z}_*^0(\phi, -\boldsymbol{\lambda}_* - \mathbf{n}_0, P_k)$ is an entire function of $\boldsymbol{\lambda}$, the result follows by analytic continuation. $\qquad \square$

Example 4.7. Let $m \geq 1$, $\lambda_1 = \ldots = \lambda_m = \lambda$, $|x|_m = \det(x'x)^{1/2}$,

$$\varphi_{\lambda,k}(x) = \frac{|x|_m^{-\lambda-n}}{\Gamma_m(-\lambda/2)} P_k(x(x'x)^{-1/2}), \qquad d_{\boldsymbol{\lambda}} = 2^{-\lambda m} \pi^{nm/2} i^{km}.$$

Then

$$(4.21) \qquad (\mathcal{F}\varphi_{\lambda,k})(y) = \frac{d_{\boldsymbol{\lambda}} \, \Gamma_m((k-\lambda)/2)}{\Gamma_m(-\lambda/2) \, \Gamma_m((\lambda+k+n)/2)} \, |y|_m^\lambda \, P_k(y(y'y)^{-1/2}).$$

If $m = 1$, then $\varphi_{\lambda,k}(x) = \frac{|x|^{-\lambda-n}}{\Gamma(-\lambda/2)} P_k\left(\frac{x}{|x|}\right)$, and we have

$$(4.22) \qquad (\mathcal{F}\varphi_{\lambda,k})(y) = \frac{d_\lambda \, \Gamma((k-\lambda)/2)}{\Gamma(-\lambda/2) \, \Gamma((\lambda+k+n)/2)} \, |y|^\lambda P_k\left(\frac{y}{|y|}\right), \qquad d_\lambda = 2^{-\lambda} \pi^{n/2} i^k.$$

5. Proofs of main results

5.1. A higher-rank analog of (1.3).

Definition 5.1. [Herz] A polynomial $P_k(x)$ on $\mathbb{R}^{n \times m}$ is called *an H-polynomial of degree k* if it is $O(m)$ right-invariant, harmonic, and determinantally homogeneous of degree k. We denote by $\mathcal{H}_k$ the space of all such polynomials.

Lemma 5.2. *Let* $P_k \in \mathcal{H}_k$,

$$(5.1) \qquad \mu_k(\boldsymbol{\lambda}) = \frac{\Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{m}_0)\,\Gamma_\Omega(\mathbf{k}_0 - \boldsymbol{\lambda}_*)}{\Gamma_\Omega(-\boldsymbol{\lambda}_*)\,\Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{k}_0 + \mathbf{n}_0)}, \qquad \boldsymbol{\lambda} \in \mathbb{C}^m.$$

If $\boldsymbol{\lambda}$ *does not belong to the polar set of* $\Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{m}_0)$, *then*

$$(5.2) \qquad (T^{\boldsymbol{\lambda}} P_k)(vr^{1/2}) = c\,\mu_k(\boldsymbol{\lambda})\,P_k(v)r^{\boldsymbol{\lambda}}, \qquad c = \pi^{m(n-m)/2}\,i^{km}\,\sigma_{m,m},$$

in the sense of S'-*distributions.*

Proof. Let us compare (4.8) and (4.19), assuming $f(v) = P_k(v)$. For all $\boldsymbol{\lambda} \in \mathbb{C}^m$,

$$(5.3) \qquad \frac{c_{\boldsymbol{\lambda}}}{\Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{m}_0)}\,(T^{\boldsymbol{\lambda}} P_k)(vr^{1/2}) = \frac{d_{\boldsymbol{\lambda}}\,\Gamma_\Omega(\mathbf{k}_0 - \boldsymbol{\lambda}_*)}{\Gamma_\Omega(-\boldsymbol{\lambda}_*)\,\Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{k}_0 + \mathbf{n}_0)}\,P_k(v)\,r^{\boldsymbol{\lambda}},$$

$$c_{\boldsymbol{\lambda}} = 2^{-|\boldsymbol{\lambda}|}\pi^{m^2/2}/\sigma_{m,m}, \qquad d_{\boldsymbol{\lambda}} = 2^{-|\boldsymbol{\lambda}|}\pi^{nm/2}i^{km},$$

in the sense of S'-*distributions. If we exclude all* $\boldsymbol{\lambda}$ *belonging to the polar set of* $\Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{m}_0)$, *we get (5.2).* $\qquad\qquad\square$

Corollary 5.3. (Cf. (1.3)) *If* $\boldsymbol{\lambda} \in \mathfrak{L}$ *(see Definition 3.1) and* $P_k \in \mathcal{H}_k$, *then*

$$(5.4) \qquad (T^{\boldsymbol{\lambda}} P_k)(v) = c\,\mu_k(\boldsymbol{\lambda})\,P_k(v), \qquad v \in V_{n,m},$$

c and $\mu_k(\boldsymbol{\lambda})$ *being the same as in (5.2).*

Proof. The function $\Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{m}_0)$ has no poles in $\mathfrak{L}$. Hence, by (5.2), we have $((T^{\boldsymbol{\lambda}} P_k)(vr^{1/2}), \phi) = c\,\mu_k(\boldsymbol{\lambda})\,(P_k(v)r^{\boldsymbol{\lambda}}, \phi)$ for all $\phi(y) \equiv \phi(vr^{1/2}) \in S(\mathbb{R}^{n \times m})$. Choose $\phi(y) = \chi(r)\,\psi(v)$, where $\chi(r)$ is a non-negative C^∞ cut-off function supported away from the boundary of Ω and $\psi(v)$ is a C^∞ function on $V_{n,m}$. By passing to polar coordinates, owing to (4.6) we obtain

$$c_\chi \int\limits_{V_{n,m}} [(T^{\boldsymbol{\lambda}} P_k)(v) - c\,\mu_k(\boldsymbol{\lambda})\,P_k(v)]\,\psi(v)\,dv = 0, \qquad c_\chi = \text{const} \neq 0.$$

This implies (5.4). $\qquad\qquad\square$

Remark 5.4. An important question is, do there exist H-polynomials of a given degree k? For $n = m$ we have exactly two such polynomials, namely, $P_0(x) \equiv 1$ and $P_1(x) = \det(x)$. It is known [Herz, p. 484] that for $2m \leq n$ there exist H-polynomials of every degree k. The space $\mathcal{H}_k$ in this case is spanned by polynomials of the form $P_k(x) = \det(a'x)^k$ where a is a complex $n \times m$ matrix satisfying $a'a = 0$ [TT, p. 27].

5.2. **Proof of Theorem 3.3.** To prove the first statement, we consider the equality

$$(5.5) \qquad ((T^{\boldsymbol{\lambda}} f)(vr^{1/2}),\ \mathcal{F}\phi) = A(\boldsymbol{\lambda})\ (r_*^{-\boldsymbol{\lambda}_* - \mathbf{n}_0}\ f(v),\ \phi),$$

$$A(\boldsymbol{\lambda}) = (2\pi)^{nm}\, \Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{m}_0)/c_{\boldsymbol{\lambda}}\, \Gamma_\Omega(-\boldsymbol{\lambda}_*),$$

which follows from (4.9) and (4.7). Suppose that $(T^{\boldsymbol{\lambda}} f)(v) = 0$ a.e. on $V_{n,m}$ for some $\boldsymbol{\lambda} \in \mathfrak{L}$. Then $(T^{\boldsymbol{\lambda}} f)(y)$, $y = vr^{1/2} \in \mathbb{R}^{n \times m}$, is zero for almost all $y \in \mathbb{R}^{n \times m}$, and (5.5) yields $A(\boldsymbol{\lambda})\ (r_*^{-\boldsymbol{\lambda}_* - \mathbf{n}_0}\, f(v),\ \phi) = 0$. The assumption (3.4) along with $\boldsymbol{\lambda} \in \mathfrak{L}$ imply $Re\, \lambda_j > j - m - 1$ and $\lambda_j \neq j - m, j - m + 2, \ldots$. Hence $\boldsymbol{\lambda}$ is not a pole of $\Gamma_\Omega(-\boldsymbol{\lambda}_*)$, and therefore $A(\boldsymbol{\lambda}) \neq 0$. This gives $(r_*^{-\boldsymbol{\lambda}_* - \mathbf{n}_0}\, f(v),\ \phi) = 0$, where the left side is understood in the sense of analytic continuation. Choosing ϕ as in the proof of Corollary 5.3, we obtain $f(v) = 0$ a.e. on $V_{n,m}$.

To prove the second statement, we note that, for $2m < n$, H-polynomials of every degree k do exist. We observe that the function $\Gamma_\Omega(\mathbf{k}_0 - \boldsymbol{\lambda}_*)$ in $\mu_k(\boldsymbol{\lambda})$ can be written as

$$\Gamma_\Omega(\mathbf{k}_0 - \boldsymbol{\lambda}_*) = \pi^{m(m-1)/4} \prod_{j=1}^{m} \Gamma\Big(\frac{k + j - \lambda_j - m}{2}\Big).$$

It has no poles in $\mathfrak{L}$ if $k > \max_j \{\lambda_j + m - j\}$. Since $\Gamma_\Omega(\boldsymbol{\lambda} + \mathbf{m}_0)$ also has no poles in $\mathfrak{L}$, then, by (5.4), $T^{\boldsymbol{\lambda}} P_k = 0$ for all such k provided that $\boldsymbol{\lambda}$ is a pole of $\Gamma_\Omega(-\boldsymbol{\lambda}_*)$ (i.e., (3.4) fails). This proves the theorem.

5.3. **Proof of Theorem 3.4.** We make use of the canonical homeomorphism of $G_{n,m}$ and $G_{n,n-m}$ under which T^λ is invariant. To be precise, given $u, v \in V_{n,m}$, let $\xi \in G_{n,m}$ and $\eta \in G_{n,m}$ be m-dimensional subspaces of $\mathbb{R}^n$ spanned u and v, respectively. We denote by $\xi^\perp \in G_{n,n-m}$ and $\eta^\perp \in G_{n,n-m}$ the corresponding orthogonal subspaces, and choose any $u_\perp \in V_{n,n-m}$ in $\xi^\perp$ and $v_\perp \in V_{n,n-m}$ in $\eta^\perp$. Since $[\eta|\xi] = [\eta^\perp|\xi^\perp]$, we can successively define the functions $F(\eta)$ on $G_{n,m}$, $F_\perp(\eta^\perp)$ on $G_{n,n-m}$, and $f_\perp(v_\perp)$ on $V_{n,n-m}$ by $F(\eta) = f(u)$, $F_\perp(\eta^\perp) = F((\eta^\perp)^\perp)$, $f_\perp(v_\perp) = F_\perp(\eta^\perp)$. Then T^λ expresses through the similar operator $T_\perp^\lambda$ on $V_{n,n-m}$ as follows:

$$(5.6) \qquad (T^\lambda f)(u) = (T_\perp^\lambda f_\perp)(u_\perp).$$

Indeed,

$$\begin{aligned}
(T^\lambda f)(u) &= \int_{G_{n,m}} F(\eta)\, [\eta|\xi]^\lambda\, d\eta = \int_{G_{n,n-m}} F_\perp(\eta^\perp)\, [\eta^\perp|\xi^\perp]^\lambda\, d\eta^\perp \\
&= \int_{V_{n,n-m}} f_\perp(v_\perp)|\det(v_\perp' u_\perp)|^\lambda\, dv_\perp = (T_\perp^\lambda f_\perp)(u_\perp).
\end{aligned}$$

If $\mathrm{rank}(G_{n,m}) = 1$, i.e., $m = 1$ or $n - 1$, the result follows from the multiplier equality (1.3) which is a particular case of (5.4).

Let $\mathrm{rank}(G_{n,m}) > 1$. Owing to (5.6), it suffices to prove the theorem for $2m \leq n$, because otherwise we have $2(n - m) < n$, and one can treat $T_\perp^\lambda$ instead of T^λ. If $\lambda_1 = \ldots = \lambda_m = \lambda$, the condition $\lambda \neq 0, 1, 2, \ldots$ coincides with (3.4). But for $2m \leq n$, the result follows from Theorem 3.3.

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DEPARTMENT OF MATHEMATICAL SCIENCES, KENT STATE UNIVERSITY, MATHEMATICS AND COMPUTER SCIENCE BUILDING, SUMMIT STREET, KENT OH 44242, USA

E-mail address: `ournyche@math.kent.edu`

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA, 70803, USA, and

INSTITUTE OF MATHEMATICS, HEBREW UNIVERSITY, JERUSALEM 91904, ISRAEL

E-mail address: `borisr@math.lsu.edu`